Iterative Methods

$$A\vec{x} = \vec{b}, \quad A = U + D + L$$

Iterative Scheme:

$$\vec{x}^{k+1} = M\vec{x}^k + \vec{f} \text{. satisfy } \vec{x}^* = M\vec{x}^* + \vec{f}$$

$$\Rightarrow \vec{e}^{k+1} = M\vec{e}^k$$

$$\Rightarrow \vec{e}^k = M^k \vec{e}_0$$

$\therefore$ The convergence of the scheme only depends on M.

Spectral radius:

$$\rho(M) := \max_i \{ |\lambda_i| \mid \lambda_i \text{ eigenvalue of } M \}.$$

Convergence: $\vec{e}^k \to \vec{0}$ $\quad \forall$ initial guess.

or $\|\vec{e}^k\| \to 0$

Thm The followings are equivalent:

(1). The scheme converges

(2). $\lim_{k\to\infty} M^k = 0$

(3). $\rho(M) < 1$.

(3) => (2).

Using Jordan Canonical Form,

$M = QJQ^{-1}$, and $M^k = QJ^kQ^{-1}$

where J is a block diagonal matrix:

$$J = \begin{bmatrix} J_{m_1}(\lambda_{i_1}) & & & \\ & J_{m_2}(\lambda_{i_2}) & & \\ & & \ddots & \\ & & & J_{m_s}(\lambda_{i_s}) \end{bmatrix}$$

$$J_{m_j}(\lambda_{i_j}) = \begin{bmatrix} \lambda_{i_j} & 1 & & \\ & \lambda_{i_j} & 1 & \\ & & \ddots & 1 \\ & & & \lambda_{i_j} \end{bmatrix} \in \mathbb{C}^{m_j \times m_j}$$

Recall multiplication of Block matrix acts same as usual matrix. (as long as each matrix multiplication make sense).

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{bmatrix}$$

$$J^2 = \begin{bmatrix} J_{m_1}(\lambda_{i_1})^2 & & & \\ & J_{m_2}(\lambda_{i_2})^2 & & \\ & & \ddots & \\ & & & J_{m_s}(\lambda_{i_s})^2 \end{bmatrix}$$

$$J_{m_j}(\lambda_{i_j})^2 = \begin{bmatrix} \lambda_{i_j} & 1 & & \\ & \lambda_{i_j} & 1 & \\ & & \ddots & 1 \\ & & & \lambda_{i_j} \end{bmatrix}^2$$

$$= \begin{bmatrix} \lambda_{i_j}^2 & 2\lambda_{i_j} & 1 & & \\ & \lambda_{i_j}^2 & 2\lambda_{i_j} & 1 & \\ & & & \ddots & 1 \\ & & & & 2\lambda_{i_j} \\ & & & & \lambda_{i_j}^2 \end{bmatrix}$$

Inductively,

$$J_{m_j}(\lambda_{i_j})^k = \begin{bmatrix} \lambda_{i_j}^k & C_1^k \lambda_{i_j}^{k-1} & \cdots & C_{m_j-1}^k \lambda_{i_j}^{k-m_j+1} \\ & \lambda_{i_j}^k & \ddots & \vdots \\ & & \ddots & C_1^k \lambda_{i_j}^{k-1} \\ & & & \lambda_{i_j}^k \end{bmatrix}$$

for $k \geq m_j - 1$

To show: all go to 0.

If $|\lambda_{ij}| < 1$, $\lambda_{ij}^{k} \to 0$

$\therefore$ diagonal is fine.

$$C_r^{k+1} / C_r^{k}$$

$$= \frac{(k+1)!}{r!\,(k+1-r)!} \cdot \frac{r!\,(k-r)!}{k!}$$

$$= \frac{k+1}{k+1-r} \to 1 \text{ as } k \to \infty .$$

$$\left(C_r^{k+1} \lambda_{ij}^{k+1-r}\right) / \left(C_r^{k+1} \lambda_{ij}^{k-r}\right)$$

$$\to \lambda_{ij} < 1 \quad \text{as} \quad k \to \infty .$$

$$\therefore \; C_r^{k} \lambda_{ij}^{k-r} \to 0 \quad \text{as} \quad k \to \infty .$$

$$\Rightarrow J_{m_j}(\lambda_{ij})^{k} \to 0 \text{ as } k \to \infty .$$

$$\Rightarrow J^{k} \to 0$$

$$\Rightarrow M^{k} = Q J^{k} Q^{-1} \to 0 .$$

Jacobi :

$$\vec{x}^{k+1} = \underbrace{D^{-1}(-U-L)}_{M_J}\vec{x}^k + D^{-1}\vec{b}.$$

Gauss Seidel :

$$\vec{x}^{k+1} = \underbrace{(L+D)^{-1}(-U)}_{M_{G\text{-}S}}\vec{x}^k + (L+D)^{-1}\vec{b}.$$

SOR :

$$\vec{x}^{k+1} = \overbrace{\underline{(L+\tfrac{1}{\omega}D)^{-1}((\tfrac{1}{\omega}-1)D-U)}}^{M_{SOR}}\vec{x}^k + (L+\tfrac{1}{\omega}D)^{-1}\vec{b}.$$

we can control $\rho(M_{SOR})$ via ω.

To check convergence,

(1) : calculate spectral radius

(2) : Gerschgorin Circle Thm for M.

(3) : check SDD of A. $(|a_{ii}| > \sum^n$

for Jacobi and G-S

Example

Iterative scheme:

$$\overrightarrow{x^{k+1}} = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \overrightarrow{x^{k}} + \vec{f}.$$

$$\det(M - \lambda I) = \lambda^2 - \lambda + \frac{6}{25} - \frac{1}{25} = 0$$

$$\Rightarrow \lambda = 0.723 \cdots \text{ or } 0.276 \cdots$$

$$\Rightarrow \rho(M) < 1$$

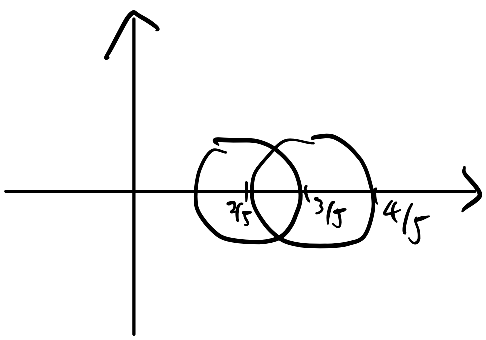
Gerschgorin Thm

$\Rightarrow$ eigenvalues lies in

$$\{ z \in \mathbb{C} \mid |z - 2/5| \leq 1/5 \}$$

$$\cup \{ z \in \mathbb{C} \mid |z - 3/5| \leq 1/5 \}$$

$$\Rightarrow \rho(M) < 1.$$



For the system:

$$A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

(3). A is SDD

$\Rightarrow$ Jacobi and G-S converge.

$$M_J = \begin{bmatrix} 0 & -2/5 \\ -3/4 & 0 \end{bmatrix}$$

(2): Gershgorin Thm

$\Rightarrow$ eigenvalues in $\{z \mid |z| \leq 3/4\}$

$\Rightarrow$ $\rho(M_J) < 1$

(1).: $\lambda^2 - \frac{3}{10} = 0$

$\Rightarrow$ $\lambda = \pm 0.5477\ldots$

$\Rightarrow$ $\rho(M_J) = 0.5477 < 1.$

$$M_{GS} = \begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & 0 \\ -3/20 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2/5 \\ 0 & 3/10 \end{bmatrix}$$

(2): Gershgorin Thm

$\Rightarrow \lambda \in \overline{B_0(2/5)} \cup \{3/10\}$.

$\Rightarrow \rho(M_{GS}) < 1$

(1): $\lambda^2 - \frac{3}{10}\lambda = 0$

$\Rightarrow \lambda = 0$ or $3/10$

$\Rightarrow \rho(M_{GS}) = 3/10 < 1$

$$A\vec{x} = \vec{b}$$

where $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$

SOR :

$$\vec{x}^{k+1} = \underbrace{(L + \tfrac{1}{\omega} D)^{-1} \left((\tfrac{1}{\omega} - 1) D - U \right)}_{M_{SOR}} \vec{x}^{k} + (L + \tfrac{1}{\omega} D)^{-1} \vec{b}.$$

$\omega = 1 \Rightarrow$ G-S.

$\omega = 0.8$,

$$M_{SOR, 0.8} = \begin{bmatrix} 1/5 & -8/25 \\ -3/25 & 49/125 \end{bmatrix}$$

$$\rho(M_{SOR, 0.8}) = 0.5142.$$

$\omega = 1.2$,

$$M_{SOR, 1.2} = \begin{bmatrix} -1/5 & -12/25 \\ 9/50 & 29/125 \end{bmatrix}$$

$$\rho(M_{SOR, 1.2}) = 0.2.$$

both converge, while $\omega = 1.2$ is better than G-S.

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & 6 \end{bmatrix}$$ from last tutorial exercise.

Not SDD, $\Rightarrow$ (3) cannot be used.

$$M_J = \begin{bmatrix} 0, & -1/4, & -1/4 \\ 2/9, & 0, & 0 \\ 0, & -4/3, & 0 \end{bmatrix}$$

$\Rightarrow \lambda \in \overline{B_0(4/3)}$

$\Rightarrow \rho(M_J) \leq 4/3$

$\Rightarrow$ (2) cannot be used.

eigenvalues: $0.38, -0.19 - 0.40i$;

$-0.19 + 0.40i$

$\rho(M_J) = 0.44 < 1$

$\therefore$ Jacobi Method converges.

Exercise

$$A\vec{x} = \vec{b}, \quad A = \begin{bmatrix} 5 & -2 \\ 1 & -2 \end{bmatrix}$$

(1). Determine convergence of Jacobi Method. Using the 3 methods.

(2). Determine convergence of G-S Method. Using the 3 methods.

(3). Determine convergence of SOR Method, for $w = 0.9,\ 1.1$.